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Fractional Effective Action at Strong Electromagnetic Fields

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In 1936, Weisskopf [K. Dan. Vidensk. Selsk. Mat. Fys. Medd. XIV (1936)] showed that for vanishing electric or magnetic fields the strong-field behavior of the one-loop Euler–Heisenberg effective Lagrangian of quantum electrodynamics (QED) is logarithmic. This result can be generalized for different limits of the Lorentz invariants. The logarithmic dependence can be interpreted as a lowest-order manifestation of an anomalous power behavior of the effective Lagrangian of QED.

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1. Introduction

In 1931 Sauter [1] and four years later Euler and Heisenberg [2] provided the first description of the vacuum properties of QED. Since then effects of QED in strong electromagnetic fields, in the order of the critical electric field $E_c = m_e^2 c^3/e\hbar$, have become a vast arena of research. Experimental verification as well as further theoretical understanding of the effects of strong-fields is still to come.

It has recently been shown that the strong-field limit of effective theories of a completely different class of quantum field theories develops anomalous power behavior. This property is also called *critical behavior* because it is experimentally observable at the critical point of second-order phase transitions. It arises in the limit of strong coupling if the beta-function (also called Gell-Mann-Low-function) has an infrared fixed point [3, 4].

In [5] a similarity between these two strong-field limits of quantum field theory has been derived. This was done by pointing out that the effective action of QED in the weak coupling expansion, the so called Euler–Heisenberg effective action, shows a power behavior typical for critical phenomena in the strong-field limit without assuming the existence of an infrared fixed point.

This fractional behavior in the weak-coupling expansion can be interpreted as a first step in the direction of a strong coupling QED, by analogy with the behavior of strong-coupling field theories mentioned above. This could be achieved by using a technique developed in the context of ϕ^4 theories, which enables one to go from a diverging weak-coupling to a converging strong-coupling expansion [3, 4, 6].

2. The Euler–Heisenberg Lagrangian

Heisenberg and Euler [2] and Weisskopf [7] derived the the nonperturbative one-loop effective action of QED for a constant classical external gauge potential A_μ, which takes the form

$$\Delta\mathcal{L}_{\text{eff}} \sim \int_0^\infty \frac{ds}{s}\Big[e^2\epsilon\beta\coth(se\epsilon)\cot(se\beta) - \frac{1}{s^2} - \frac{e^2}{3}\left(\epsilon^2-\beta^2\right)\Big]\mathrm{e}^{-\mathrm{i}s(m_e^2-\mathrm{i}\eta)} \quad (1)$$

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where the Lorentz-invariant variables ϵ and β are defined as

$$\epsilon^2-\beta^2 :=\vec{E}^2-\vec{B}^2 := -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} =: 2S, \quad (2)$$

$$\epsilon\beta :=\vec{E}\cdot\vec{B} := -F_{\mu\nu}\tilde{F}^{\mu\nu} =: P. \quad (3)$$

Here we used the electric and magnetic field strength $\vec{E}$ and $\vec{B}$ the field strength tensor $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and the scalar and pseudo-scalar invariants S and P. The explicit form of the quantities ϵ and β is

$$\epsilon=\sqrt{\sqrt{S^2+P^2}+S},\ \beta=\sqrt{\sqrt{S^2+P^2}-S}. \quad (4)$$

3. Reformulation of the Euler–Heisenberg Lagrangian

Using a sum expansion for $\coth(x)\cot(x)$ which was first derived in [8] and performing a contour integral the real part of the Euler–Heisenberg effective action (1) can be brought into the form [5]

$$\Delta\mathcal{L}^{\Re}_{\text{eff}} =\frac{e^2}{(2\pi)^3}\sum_{n=1}^{\infty}\frac{\beta\epsilon}{n}\left[\coth\left(\pi\frac{\epsilon}{\beta}n\right)J\left(\frac{in\pi E_c}{\beta}\right)\right. \\ \left. -\coth\left(\pi\frac{\beta}{\epsilon}n\right)J\left(\frac{n\pi E_c}{\epsilon}\right)\right], \quad (5)$$

with

$$J(z)=-\left(\mathrm{e}^{-z}\mathrm{Ei}(z)+\mathrm{e}^{z}\mathrm{Ei}(-z)\right), \\ \mathrm{Ei}(x)=-\int_{-x}^{\infty}\frac{\mathrm{e}^{-t}}{t}\,dt \quad (6)$$

where $\mathrm{Ei}(x)$ denotes the Exponential integral.

4. Strong-field approximation

One can now use the sum formulation (5) of the Euler–Heisenberg Lagrangian to look for the leading order of the strong-field limit ($|\vec{E}|, |\vec{B}|\gg E_c$). We must investigate this limit in terms of the Lorentz invariant terms S and P (or equivalently ϵ and β) since the Lagrangian is formulated in terms of them. Because there is no direct way to translate the above limit to this variables we have to limit us to some special cases.

4.1. Small-P expansion

The case $|S/P|\gg 1$, for which the component of the magnetic field in direction of the magnetic field is small, was studied in [9]. For this case it is sufficient to Taylor-expand the Lagrangian around $P\sim 0$ and take only the first term $P=0$ into account. This corresponds to either $\epsilon=0$ or $\beta=0$. Inserting this in (5) and taking only the leading logarithmic term into account the real part of the Lagrangian takes the form [5]

$$\mathcal{L}^{\Re} =\frac{1}{2}(\vec{E}^2-\vec{B}^2)+\frac{e^2}{24\pi^2}(\vec{E}^2-\vec{B}^2) \\ \times\log\left(\frac{|\vec{E}^2-\vec{B}^2|}{E_c^2}\right)+\mathcal{O}\left(\frac{S}{E_c^2},\frac{P}{S}\right). \quad (7)$$

This can be brought into the form

$$\mathcal{L}^{\Re} =\frac{1}{2}E_c^{-\delta}(\vec{E}^2-\vec{B}^2)|\vec{E}^2-\vec{B}^2|^{\delta/2}+\dots \quad (8)$$

where we define the anomalous power

$$\delta := \frac{e^2}{12\pi}. \quad (9)$$

4.2. General strong-field case ($\beta, \epsilon\gg E_c$)

By investigating the general case $\beta, \epsilon\gg E_c$ one finds, that one has to limit oneself to the case that ϵ and β are of the same order of magnitude, i.e. $\epsilon/\beta\sim\mathcal{O}(1)$, to get analytic results. By doing so and using a property of the Dedekind eta function which arises from reorganizing the sums during the strong-field approximation one finds for the total effective Lagrangian [5]

$$\mathcal{L}^{\Re} =\frac{1}{2}(\epsilon^2-\beta^2)+\frac{e^2}{24\pi^2}\left[\epsilon^2\log\left(\frac{|\beta|}{E_c}\right)\right. \\ \left. -\beta^2\log\left(\frac{|\epsilon|}{E_c}\right)\right]+\mathcal{O}\left(\frac{\epsilon^2}{E_c^2},\frac{\beta^2}{E_c^2}\right). \quad (10)$$

This is formulated with an anomalous power as

$$\mathcal{L}^{\Re} = \frac{1}{2}E_c^{-\delta}\left(\epsilon^2|\beta|^\delta - \beta^2|\epsilon|^\delta\right) + \dots \tag{11}$$

where the coefficient $\delta = e^2/12\pi$ is the same as in the small-P case.

4.3. Small-S expansion

A special case of the above result is the case for which $|S/P| \gg 1$, since $\epsilon, \beta \gg 1$ implies that $|P| > E_c^2$ but $\epsilon/\beta \sim \mathcal{O}(1)$ does not necessarily mean that S is small.

By expanding ϵ and β for $S/|P|$ around 0 in (11) one finds [5]

$$\mathcal{L}^{\Re} = \frac{1}{2}(\vec{E}^2 - \vec{B}^2) + \frac{e^2}{48\pi^2}(\vec{E}^2 - \vec{B}^2)\log\left(\frac{|\vec{E}\cdot\vec{B}|}{E_c^2}\right) + \mathcal{O}\left(\frac{S}{E_c^2}, \frac{S^2}{P^2}\right), \tag{12}$$

which can be brought in the form

$$\mathcal{L}^{\Re} = \frac{1}{2}E_c^{-\delta}(\vec{E}^2 - \vec{B}^2)|\vec{E}\cdot\vec{B}|^{\delta/2} + \dots \tag{13}$$

with the anomalous power $\delta = e^2/12\pi$.

5. Conclusions

It has been demonstrated that the leading order of the Lagrangian of QED in the limit of strong electromagnetic fields is logarithmic and can be written in the fractional form for the three different cases:

1. $|S/P| \gg 1$,
2. ϵ, $\beta \gg E_c$ and $\epsilon/\beta \sim \mathcal{O}(1)$,
3. $|P/S| \gg 1$.

However we were not able to derive a result for the case $S, P \gg E_c^2$, which is equivalent to $|\vec{E}| \gg |\vec{B}| \gg E_c$ or $|\vec{B}| \gg |\vec{E}| \gg E_c$ while the fields are almost parallel. Combining the results of the cases 1 and 3 we can conjecture the more general result

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}E_c^{-2\delta}(\vec{E}^2 - \vec{B}^2)\Big(|\vec{E}^2 - \vec{B}^2||\vec{E}\cdot\vec{B}|\Big)^{\delta/2} + \dots, \tag{14}$$

with the anomalous power $\delta = e^2/(12\pi)$. The above formulation includes the cases 1 and 3 in there respective limits and thus presents a more general result.

As has been shown in [5] also scalar QED shows the same fractional behavior for strong electromagnetic fields but with the anomalous power $\delta_s = \delta/4 = e^2/(48\pi)$.

Equation (14) provides a fractional formulation of QED which exhibits interesting similarities to the fractional quantum field theories discussed in the introduction. Using this analogy it might be possible to derive a converging strong coupling series for QED using the methods of [3, 4, 6].

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